

NATURAL CONVECTION OF A CONDUCTING LIQUID IN A SPHERICAL LAYER.

II. METHOD OF SOLUTION. CALCULATED RESULTS

S. V. Solov'ev, V. K. Bulgakov, and
S. V. Kuznetsov

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A system of differential equations describing the natural convection of the conducting liquid in the earth's core is solved numerically by the methods of a reference volume and finite differences.

The system of differential equations (7)-(11) was solved numerically by the method of a reference volume [1], which is the most efficient in solving problems of heat transfer and liquid dynamics, and the method of finite differences.

After integration of Eqs. (7)-(11) over the reference volume (Fig. 1) we obtain their discrete analogs: for the energy equation

$$a_p \delta_p = a_E \delta_E + a_W \delta_W + a_N \delta_N + a_S \delta_S + B, \quad (12)$$

where

$$a_p = \frac{\Delta r \Delta \theta}{\text{Sh } \Delta \tau} + \frac{C_1}{r_p^2 \sin \theta_p} \left[\frac{(\Psi_{ne} - \Psi_{se}) + |\Psi_{ne} - \Psi_{se}|}{2} - \frac{(\Psi_{nw} - \Psi_{sw}) + |\Psi_{nw} - \Psi_{sw}|}{2} - \frac{(\Psi_{ne} - \Psi_{nw}) + |\Psi_{ne} - \Psi_{nw}|}{2} + \frac{(\Psi_{se} - \Psi_{sw}) + |\Psi_{se} - \Psi_{sw}|}{2} \right] + \frac{1}{\text{Pe } r_p^2} \left[\frac{r_e^2 \Delta \theta}{\Delta r} + \frac{r_w^2 \Delta \theta}{\Delta r} + \frac{\Delta r \sin \theta_n}{\Delta \theta \sin \theta_p} + \frac{\Delta r \sin \theta_s}{\Delta \theta \sin \theta_p} \right];$$

$$a_E = \frac{r_e^2 \Delta \theta}{\text{Pe } r_p^2 \Delta r} - \frac{C_1}{r_p^2 \sin \theta_p} \left[\frac{(\Psi_{ne} - \Psi_{se}) - |\Psi_{ne} - \Psi_{se}|}{2} \right];$$

$$a_W = \frac{r_w^2 \Delta \theta}{\text{Pe } r_p^2 \Delta r} + \frac{C_1}{r_p^2 \sin \theta_p} \left[\frac{(\Psi_{nw} - \Psi_{sw}) - |\Psi_{nw} - \Psi_{sw}|}{2} \right];$$

$$a_N = \frac{\Delta r \sin \theta_n}{\text{Pe } r_p^2 \Delta \theta \sin \theta_p} + \frac{C_1}{r_p^2 \sin \theta_p} \left[\frac{(\Psi_{ne} - \Psi_{nw}) - |\Psi_{ne} - \Psi_{nw}|}{2} \right];$$

$$a_S = \frac{\Delta r \sin \theta_s}{\text{Pe } r_p^2 \Delta \theta \sin \theta_p} - \frac{C_1}{r_p^2 \sin \theta_p} \left[\frac{(\Psi_{se} - \Psi_{sw}) + |\Psi_{se} - \Psi_{sw}|}{2} \right];$$

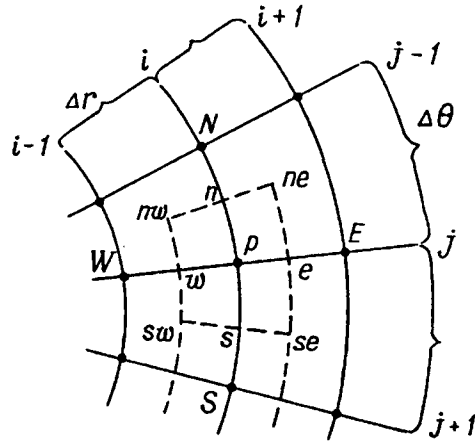


Fig. 1. Reference volume.

$$B = \frac{\vartheta_p^0 \Delta r \Delta \theta}{\text{Sh } \Delta \tau};$$

for the vorticity equation

$$a_p \omega_p = a_E \omega_E + a_W \omega_W + a_N \omega_N + a_S \omega_S + B, \quad (13)$$

where

$$\begin{aligned} a_p = & \frac{\Delta r \Delta \theta}{\text{Sh } \Delta \tau} + \frac{C_1}{r_p} \left[\frac{1}{r_e \sin \theta_p} \left(\frac{(\Psi_{ne} - \Psi_{se}) + |\Psi_{ne} - \Psi_{se}|}{2} \right) - \right. \\ & - \frac{1}{r_w \sin \theta_p} \left(\frac{(\Psi_{nw} - \Psi_{sw}) - |\Psi_{nw} - \Psi_{sw}|}{2} \right) - \\ & - \frac{1}{r_p \sin \theta_n} \left(\frac{(\Psi_{ne} - \Psi_{nw}) + |\Psi_{ne} - \Psi_{nw}|}{2} \right) + \\ & \left. + \frac{1}{r_p \sin \theta_s} \left(\frac{(\Psi_{se} - \Psi_{sw}) - |\Psi_{se} - \Psi_{sw}|}{2} \right) \right] + \\ & + \frac{1}{\text{Re } r_p^2} \left[\frac{r_e^2 \Delta \theta}{\Delta r} + \frac{r_w^2 \Delta \theta}{\Delta r} + \frac{\Delta r \sin \theta_n}{\Delta \theta \sin \theta_p} + \frac{\Delta r \sin \theta_s}{\Delta \theta \sin \theta_p} \right]; \\ a_E = & \frac{r_e^2 \Delta \theta}{\text{Re } r_p^2 \Delta r} - \frac{C_1}{r_e r_p \sin \theta_p} \left(\frac{(\Psi_{ne} - \Psi_{se}) - |\Psi_{ne} - \Psi_{se}|}{2} \right); \\ a_W = & \frac{r_w^2 \Delta \theta}{\text{Re } r_p^2 \Delta r} + \frac{C_1}{r_w r_p \sin \theta_p} \left(\frac{(\Psi_{nw} - \Psi_{sw}) + |\Psi_{nw} - \Psi_{sw}|}{2} \right); \\ a_N = & \frac{\Delta r \sin \theta_n}{\text{Re } r_p^2 \Delta \theta \sin \theta_p} + \frac{C_1}{r_p^2 \sin \theta_n} \left(\frac{(\Psi_{ne} - \Psi_{nw}) - |\Psi_{ne} - \Psi_{nw}|}{2} \right); \end{aligned}$$

$$\begin{aligned}
a_S &= \frac{\Delta r \sin \theta_s}{\operatorname{Re} r_p^2 \Delta \theta \sin \theta_p} - \frac{C_1}{r_p^2 \sin \theta_s} \left(\frac{(\Psi_{se} - \Psi_{sw}) - |\Psi_{se} - \Psi_{sw}|}{2} \right); \\
B &= \frac{w_p^0 \Delta r \Delta \theta}{\operatorname{Sh} \Delta \tau} - \frac{w_p^0}{\operatorname{Re}} \left(\frac{1}{r_w} - \frac{1}{r_e} \right) (\operatorname{ctan} \theta_s - \operatorname{ctan} \theta_n) - \frac{C_2 \operatorname{Gr} \Delta r (\vartheta_n - \vartheta_s)}{2 \operatorname{Re}^2 r_p} \\
&+ \frac{SC_3}{\operatorname{Re}_m} \left[B_{r_p} \left(\frac{B_{\theta_e} - 2B_{\theta_p} + B_{\theta_w}}{\Delta r} + \frac{B_{\theta_e} + B_{\theta_p}}{r_e} - \frac{B_{\theta_p} + B_{\theta_w}}{r_w} - \right. \right. \\
&\quad \left. \left. - \frac{B_{r_n} + B_{r_{ne}} - B_{r_s} - B_{r_{se}}}{4r_e \Delta \theta} + \frac{B_{r_{nw}} + B_{r_n} - B_{r_s} - B_{r_{sw}}}{4r_w \Delta \theta} \right) \Delta \theta + \right. \\
&+ B_{\theta_p} \Delta \theta \left(\frac{B_{r_e} + B_{r_p}}{2r_e} - \frac{B_{r_p} + B_{r_w}}{2r_w} \right) + B_{\theta_p} \left(\frac{B_{\theta_{ne}} + B_{\theta_e} - B_{\theta_{nw}} - B_{\theta_w}}{4r_p \Delta r} - \right. \\
&\quad \left. - \frac{B_{\theta_e} + B_{\theta_{se}} - B_{\theta_w} - B_{\theta_{sw}}}{4r_p \Delta r} + \frac{B_{\theta_n} - B_{\theta_s}}{r_p^2} - \frac{B_{r_n} - 2B_{r_p} + B_{r_s}}{r_p^2 \Delta \theta} \right) \Delta r + \\
&\quad \left. + \left(\frac{B_{\theta_e} - B_{\theta_w}}{2} \Delta \theta - \frac{B_{r_n} - B_{r_s}}{2r_p} \Delta r \right) \left(\frac{B_{r_e} - B_{r_w}}{2\Delta r} + \frac{B_{\theta_n} - B_{\theta_s}}{2r_p \Delta \theta} \right) \right];
\end{aligned}$$

for the stream function

$$a_p \Psi_p = a_E \Psi_E + a_W \Psi_W + a_N \Psi_N + a_S \Psi_S + B, \quad (14)$$

where

$$a_E = \frac{\Delta \theta}{\Delta r}; \quad a_W = \frac{\Delta \theta}{\Delta r}; \quad a_N = \left(\frac{1}{\Delta \theta} - \frac{\operatorname{ctan} \theta_n}{2} \right) \frac{\Delta r}{r_p^2}; \quad a_S = \left(\frac{1}{\Delta \theta} + \frac{\operatorname{ctan} \theta_s}{2} \right) \frac{\Delta r}{r_p^2};$$

$$a_p = a_E + a_W + a_N + a_S; \quad B = \omega_p r_p \Delta r \Delta \theta \sin \theta_p;$$

for the equation of radial magnetic induction

$$a_p B_{r_p} = a_E B_{r_E} + a_W B_{r_W} + a_N B_{r_N} + a_S B_{r_S} + B, \quad (15)$$

where

$$\begin{aligned}
a_p &= \frac{1}{\operatorname{Re}_m} \left[\frac{r_e^2 \Delta \theta}{r_p^2 \Delta r} + \frac{r_w^2 \Delta \theta}{r_p^2 \Delta r} + \frac{\Delta r \sin \theta_n}{r_p^2 \Delta \theta \sin \theta_p} + \frac{\Delta r \sin \theta_s}{r_p^2 \Delta \theta \sin \theta_p} \right] - \\
&- \frac{1}{r_p^2 \sin \theta_p} \left[\frac{(\Psi_{ne} + \Psi_e - \Psi_{nw} - \Psi_w)}{4} - \frac{(\Psi_{se} + \Psi_e - \Psi_{sw} - \Psi_w)}{4} \right] + \frac{\Delta r \Delta \theta}{\operatorname{Sh} \Delta \tau};
\end{aligned}$$

$$a_E = \frac{r_e^2 \Delta \theta}{\operatorname{Re}_m r_p^2 \Delta r}; \quad a_W = \frac{r_w^2 \Delta \theta}{\operatorname{Re}_m r_p^2 \Delta r};$$

$$\begin{aligned}
a_N &= \frac{\Delta r \sin \theta_n}{\text{Re}_m \Delta \theta r_p^2 \sin \theta_p} + \frac{(\Psi_{ne} + \Psi_e - \Psi_{nw} - \Psi_w)}{4r_p^2 \sin \theta_p} \Delta r; \\
a_S &= \frac{\Delta r \sin \theta_s}{\text{Re}_m \Delta \theta r_p^2 \sin \theta_p} - \frac{(\Psi_{se} + \Psi_e - \Psi_{sw} - \Psi_w)}{4r_p^2 \sin \theta_p} \Delta r; \\
B &= \frac{B_{r_p}^0 \Delta r \Delta \theta}{\text{Sh } \Delta \tau} + \frac{\Delta r}{r_p^2 \sin \theta_p} \left(\frac{(\Psi_n - \Psi_p)(B_{\theta_n} + B_{\theta_p}) - (\Psi_p - \Psi_s)(B_{\theta_s} + B_{\theta_p})}{r_p \Delta \theta} \right) - \\
&\quad - \frac{1}{\text{Re}_m} \left[\frac{2\Delta r \Delta \theta B_{r_p}^0}{r_p^2} + \frac{2\Delta r \Delta \theta B_{\theta_p} \text{ctan } \theta_p}{r_p^2} + \frac{\Delta r (B_{\theta_n} - B_{\theta_s})}{r_p^2} \right];
\end{aligned}$$

for the equation of meridian magnetic induction

$$a_p B_{\theta_p} = a_E B_{\theta_E} + a_W B_{\theta_W} + a_N B_{\theta_N} + a_S B_{\theta_S} + B, \quad (16)$$

where

$$\begin{aligned}
a_p &= \frac{1}{\text{Re}_m} \left[\frac{r_e^2 \Delta \theta}{r_p^2 \Delta r} + \frac{r_w^2 \Delta \theta}{r_p^2 \Delta r} + \frac{\Delta r \sin \theta_n}{r_p^2 \Delta \theta \sin \theta_p} + \frac{\Delta r \sin \theta_s}{r_p^2 \Delta \theta \sin \theta_p} \right] - \\
&\quad - \frac{\Delta \theta}{r_p \sin \theta_p} \left[\frac{(\Psi_n - \Psi_s) \Delta r}{4r_p^2} - \frac{(\Psi_n + \Psi_{ne} - \Psi_s - \Psi_{se})}{4r_e \Delta \theta} + \right. \\
&\quad \left. + \frac{(\Psi_{nw} + \Psi_n - \Psi_s - \Psi_{sw})}{4r_w \Delta \theta} \right] + \frac{\Delta r \Delta \theta}{\text{Sh } \Delta \tau}; \\
a_E &= \frac{r_e^2 \Delta \theta}{\text{Re}_m r_p^2 \Delta r} - \frac{\Psi_n + \Psi_{ne} - \Psi_s - \Psi_{se}}{4r_e r_p \sin \theta_p}; \quad a_W = \frac{r_w^2 \Delta \theta}{\text{Re}_m r_p^2 \Delta r} + \frac{\Psi_{nw} + \Psi_n - \Psi_s - \Psi_{sw}}{4r_w r_p \sin \theta_p}; \\
a_N &= \frac{\Delta r \sin \theta_n}{\text{Re}_m \Delta \theta r_p^2 \sin \theta_p} + \frac{\Delta r (\Psi_n + \Psi_p)}{4r_p^3 \sin \theta_p}; \quad a_S = \frac{\Delta r \sin \theta_s}{\text{Re}_m \Delta \theta r_p^2 \sin \theta_p} - \frac{\Delta r (\Psi_p + \Psi_s)}{4r_p^3 \sin \theta_p}; \\
B &= \frac{B_{\theta_p}^0 \Delta r \Delta \theta}{\text{Sh } \Delta \tau} + \frac{1}{\text{Re}_m} \left(\frac{(B_{r_n} - B_{r_s}) \Delta r}{r_p^2} - \frac{B_{\theta_p}^0 \Delta r \Delta \theta}{r_p^2 \sin^2 \theta_p} \right) - \\
&\quad - \frac{\Delta \theta}{r_p \sin \theta_p} \left(\frac{(B_{r_e} + B_{r_p})(\Psi_e - \Psi_p) - (B_{r_p} + B_{r_w})(\Psi_p - \Psi_w)}{\Delta r} \right).
\end{aligned}$$

In discrete form the boundary conditions (BC) for Eqs. (12)-(16) are written as follows. For the stream function:

a) on the walls

$$\Psi_{0j} = \Psi_{Kj} = 0; \quad \Psi_{lj} = \Psi_{0j}; \quad \Psi_{Kj} = \Psi_{K-1j}; \quad j = \overline{0, M};$$

b) on the symmetry axis

$$\Psi_{i0} = \Psi_{iM} = 0;$$

$$\Psi_{i-1} - 2\Psi_{i0} + \Psi_{i1} = \Psi_{iM+1} - 2\Psi_{iM} + \Psi_{iM-1} = 0; \quad i = \overline{0, K}.$$

For the vorticity:

a) on the walls

$$\omega_{0j} = -\frac{3(\Psi_{1j} - \Psi_{0j})}{\sin \theta_j (\Delta r)^2} + \frac{\omega_{1j}}{2};$$

$$\omega_{Kj} = -\frac{3(\Psi_{K-1j} - \Psi_{Kj}) r_1}{r_2 \sin \theta_j (\Delta r)^2} + \frac{\omega_{K-1j}}{2}; \quad j = \overline{0, M};$$

b) on the symmetry axis

$$\omega_{i0} = \omega_{iM} = 0; \quad i = \overline{1, K}.$$

For the temperature:

a) on the walls

$$\vartheta_{0j} = 1; \quad \vartheta_{Kj} = 0; \quad j = \overline{0, M} \quad (\text{BC of the first kind});$$

$$\frac{4\vartheta_{1j} - 3\vartheta_{0j} - \vartheta_{2j}}{2\Delta r} = 1 \quad (\text{BC of the second kind});$$

$$\frac{4\vartheta_{1j} - 3\vartheta_{0j} - \vartheta_{2j}}{2\Delta r} = \text{Bi}_1 \vartheta_{0j}; \quad \frac{\vartheta_{K-2j} + 3\vartheta_{Kj} - 4\vartheta_{K-1j}}{2\Delta r} = \text{Bi}_2 \vartheta_{Kj} \quad (\text{BC of the third kind});$$

b) on the symmetry axis

$$\vartheta_{i1} = \vartheta_{i0}; \quad \vartheta_{iM} = \vartheta_{iM-1}; \quad i = \overline{0, K}.$$

For the magnetic induction:

a) on the walls

$$B_{r_{0j}} = \text{const}; \quad B_{\theta_{0j}} = \text{const}; \quad B_{r_{Kj}} = \text{const}; \quad B_{\theta_{Kj}} = \text{const}; \quad j = \overline{0, M};$$

b) on the symmetry axis

$$B_{r_{i1}} = B_{r_{i0}}; \quad B_{r_{iM}} = B_{r_{iM-1}}; \quad B_{\theta_{i1}} = B_{\theta_{i0}}; \quad B_{\theta_{iM}} = B_{\theta_{iM-1}}; \quad i = \overline{0, K}.$$

Here, the following notation is adopted: $f(r, \theta) = f(r_i, \theta_j) = f_{ij}$, where f is some function.

To solve Eqs. (12)-(16), the Gauss-Siedel iteration technique with the use of lower relaxation was adopted. In implementing the iteration technique it was noticed that if calculations of iterations were alternated in the $0-\pi$ and $\pi-0$ directions, then the time of calculation up to obtaining a physical solution with a prescribed degree of accuracy was considerably reduced. Calculations were performed on a 15 (for the radius) \times 25 (for the angle) grid. As a result of the calculations, the temperature and stream-function fields and the radial and meridian components of the velocity and the magnetic induction were obtained.

Figures 2, 3 provide calculated results for steady-state conditions for $r_2/r_1 = 1/2.62$, $\text{Gr}/\text{Re}^2 = 10^3$, $S/\text{Re}_m = 1$ [2]. On the surfaces of the spheres the temperatures were assumed to be constant (boundary conditions of the first kind) but the inner sphere was more heated. The boundary conditions for the magnetic induction at the boundaries Γ_1, Γ_2 were as follows (Fig. 3a and b, respectively):

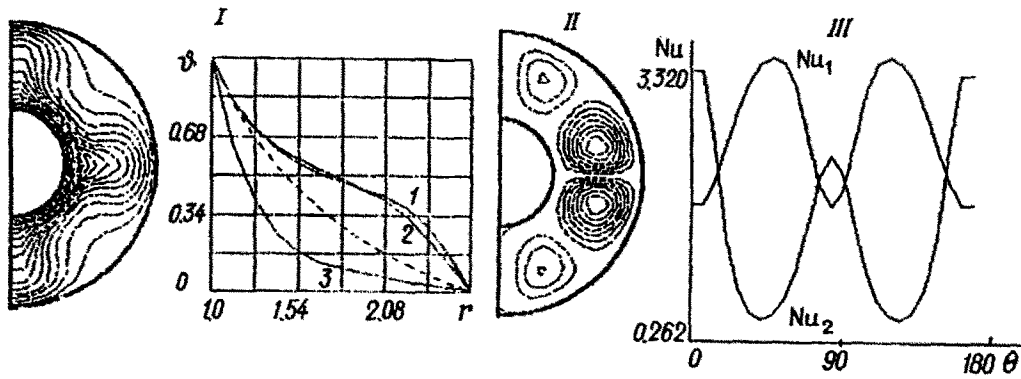


Fig. 2. Calculated results for convective heat transfer of a nonconducting liquid: I) temperature: 1) $\theta = 0, 180$; 2) 90 ; 3) $45, 135^\circ$; II) stream function $|\Psi_{\max}| \sim 5.20$; III) local Nusselt numbers: $\overline{Nu}_{r_1} = 2.629$; $\overline{Nu}_{r_2} = 1.031$.

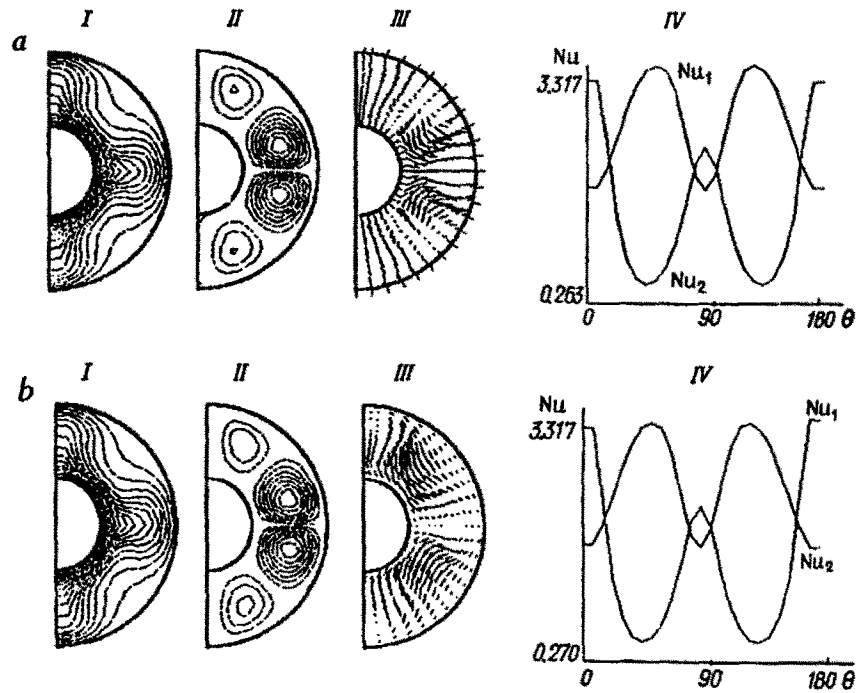


Fig. 3. Convection of the conducting liquid in the earth's core (at the core boundaries the radial (a) and meridian (b) components of the magnetic induction are prescribed): I) temperature, II) stream function [a) $|\Psi_{\max}| \sim 5.16$, b) 4.99]; III) magnetic induction [a) $B_r^{\max} \sim 2.71$, $|B_\theta^{\max}| \sim 3$; b) $6.92, 9.74$]; IV) local Nusselt numbers [a) $\overline{Nu}_{r_1} = 2.621$, $\overline{Nu}_{r_2} = 1.028$; b) $2.572, 1.007$].

$$B_{r_i} = 1; B_{\theta_i} = 0 \text{ and } B_{r_i} = 0; B_{\theta_i} = 1, i = 1, 2.$$

A comparison of the results obtained reveals that account for the magnetic induction influences the temperature field insignificantly. This is confirmed by the distribution of the local Nusselt numbers Nu_1, Nu_2 and their averaged values $\overline{Nu}_{r_1}, \overline{Nu}_{r_2}$. The flow pattern also does not change qualitatively although the intensity of the convection decreases in the case of account for the magnetic induction. The maximum value of the stream function (Fig. 2), corresponding to the natural-convection heat transfer of a nonconducting liquid, is 5.20. In the case of a conducting liquid account for the magnetic conduction leads to suppression of convection in the core, and this effect is most pronounced in prescribing the meridian component of the magnetic induction B_θ (Fig. 3b). Here $B_r^{\max} \sim 6.92$, $|B_\theta^{\max}| \sim 9.74$, $|\Psi_{\max}| \sim 4.99$ as compared to the case of radial magnetic induction (Fig. 3a), where

$B_r^{\max} \sim 2.71$, $|B_\theta^{\max}| \sim 3$, $|\Psi_{\max}| \sim 5.16$. The analogous conclusion that the magnetic field prevents initiation of thermal convection in the earth's core is drawn in [3].

Comparing the magnetic-induction fields in Fig. 3, one can notice that prescribing the radial component of magnetic induction at the core boundaries (Fig. 3a) results in initiation of the meridian component. Here, $|B_\theta^{\max}|/B_r^{\max} \sim 1.1 > 1$. In the case of prescribing the meridian component of magnetic induction at the core boundaries the radial component appears (Fig. 3b); however, here $B_r^{\max}/|B_\theta^{\max}| \sim 0.71 < 1$. For all the calculated fields (temperature, stream function, magnetic induction, local Nusselt numbers) symmetry relative to the angle $\theta = 90^\circ$ occurs.

As is seen from the figures, there is four-cell flow in the liquid layer. Its intensity is considerably smaller at the poles ($\theta = 0, 180^\circ$) than in the central region.

Figure 2 shows a temperature distribution over the thickness of the spherical layer typical for natural-convection heat transfer (the dashed line is the temperature distribution in the case of pure thermal conductivity). The change in the local Nusselt numbers has a "wavy" character, it copies the temperature profile. Testing with the aid of the heat-balance equation

$$\overline{Nu}_{r_1} = R_2 \overline{Nu}_{r_2}$$

showed that the calculated \overline{Nu}_{r_1} and \overline{Nu}_{r_2} are in good agreement.

The averaged Nusselt numbers have the tendency to decrease in the case of convection of a conducting liquid as compared to convection of a nonconducting liquid.

Thus, the mathematical model suggested gives the possibility of calculating the fields of the stream function, temperature, magnetic induction, and local and averaged Nusselt numbers. An analysis of the results obtained allows the following conclusion to be drawn – natural convection exerts an influence on the magnetic field of the earth's core, thus creating conditions for its generation.

NOTATION

Ψ , stream function; ω , vorticity; ϑ , temperature; B_r , B_θ , radial and meridian components of magnetic induction; θ , polar angle; r , current radius; Δr , radius step; $\Delta\theta$, angle step; $\Delta\tau$, time step; Sh, Pr, Re, Gr, dimensionless Strouhal, Peclet, Reynolds, and Grashof numbers, respectively; S , Re_m , magnetic-interaction parameter and magnetic Reynolds number, respectively; Bi_1 , Bi_2 , Biot numbers at the inner and outer boundaries of the core; Γ_1 , Γ_2 , boundaries of the inner and outer spheres; $R_2 = r_2/r_1$, dimensionless outer radius of the core; r_2 , r_1 , outer and inner dimensional radii of the core; Nu_1 , Nu_2 , local Nusselt numbers on the internal (Γ_1) and external (Γ_2) surfaces of the core; \overline{Nu}_{r_1} , \overline{Nu}_{r_2} , averaged Nusselt numbers on the internal and external surfaces of the core; C_1 , constant that is equal to unity with account for the convection terms in the energy and vorticity equations and is equal to zero when the convection terms in these equations are neglected; C_2 , constant that is equal to unity with account for the lifting forces in the vorticity equation and is equal to zero when the lifting forces are neglected; C_3 , constant that is equal to unity with account for the magnetic forces in the vorticity equation and is equal to zero when the magnetic forces are neglected. Subscripts and superscripts: 1, 2, internal and external surface; m, magnetic; E , W , N , S , p , e , w , n , s , ne , nw , se , sw , nodal points of the fragment of the reference volume; max, maximum; r , radial component; θ , meridian component.

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